

Husimi-Wigner representation of chaotic eigenstates

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Just as a coherent state may be considered as a quantum point, its restriction to a factor space of the full Hilbert space can be interpreted as a quantum plane. The overlap of such a factor coherent state with a full pure state is akin to a quantum section. It defines a reduced pure state in the cofactor Hilbert space. The collection of all the Wigner functions corresponding to a full set of parallel quantum sections defines the Husimi-Wigner representation. It occupies an intermediate ground between drastic suppression of nonclassical features, characteristic of Husimi functions, and the daunting complexity of higher dimensional Wigner functions. After analysing these features for simpler states, we exploit this new representation as a probe of numerically computed eigenstates of chaotic Hamiltonians. The individual two-dimensional Wigner functions resemble those of semiclassically quantized states, but the regular ring pattern is broken by dislocations.

Keywords: phase space representations, Wigner function, Husimi function, semiclassical mechanics, chaotic eigenstates.

1. Introduction

It is well known that phase space representations of quantum mechanics are powerful tools for studying the correspondence between the density operator and classical distributions in phase space. The several choices of representation are partially distinguished by the different ways they highlight classical structures against a background of quantum interferences. In the case of the Wigner function (Wigner 1932), $W(\mathbf{x})$, in terms of the phase space variables, $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_L) = (p_1, \dots, p_L, q_1, \dots, q_L)$, the oscillations due to interferences may even have higher amplitudes than the classical region. In contrast, the Husimi function (Husimi 1940, Takashi 1986), which

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may be defined as a coarse-graining of the Wigner function, [†]

$$\mathcal{H}(\mathbf{X}) = (\pi\hbar)^{-L} \int d\mathbf{x} W(\mathbf{x}) \exp \frac{-(\mathbf{x} - \mathbf{X})^2}{\hbar}, \quad (1.1)$$

subtly disguises information on quantum coherences to the point that they may be numerically undetectable, while clearly displaying most classical structures (Toscano & Ozorio de Almeida 1999).

In the case of $(2L)$ -dimensional phase spaces ($2L$ -D), with $L > 1$, the approximate classical support for a quantum state may take the form of a discrete set of points. These correspond to either a coherent state, squeezed or not, or their superposition, sometimes known as *Schrödinger cat states*. Alternatively, semiclassical *Van Vleck states* (Van Vleck 1999) correspond to L -D (Lagrangian) surfaces. Of even higher dimension is the support of *ergodic states* satisfying Schnirelman's theorem (Schnirelman 1974, Colin de Verdière 1985 & Zelditch 1987): eigenstates of (classically) chaotic Hamiltonians, supported by the full $(2L - 1)$ -D energy shell. Of course, all these types of state can be superposed in various ways, which in their turn produce new interferences.

Even though we cannot directly visualize such classical structures in a higher dimensional phase space, they will show up in appropriate 2-D sections of the corresponding Husimi function. Even so, it will be virtually impossible to extricate the crucial quantum phase information in this representation, unless all analytical properties concerning the state are known (Leboeuf & Voros 1990 and Leboeuf & Voros 1995). The situation for the Wigner function is just the opposite: all phase information is immediately available in the oscillatory interference pattern, but it becomes hard to sort out its embarrassing richness. For example, a 2-D section may contain a (plain) periodic orbit. States that are *scarred* by this periodic orbit (Heller 1984, Bogomolny 1988 & Berry 1989) are clearly distinguishable in the section of the Wigner function through this plane (Toscano *et al.* 2001), however, interferences also arise which can only be generated by classical structures that are nowhere near this 2-D plane. There is certainly need for a more manageable representation of the interference effects that decorate classical structures of general pure states. Indeed, the Wigner function itself, in the simple case that $L = 1$, is a good example of a comprehensible interference pattern (Berry 1977a).

Our purpose here is to develop a new tool for the analysis of the chaotic eigenstates of higher dimensional systems. Even though there has been continuing interest in their localization and statistical properties, starting with references Voros 1976 and Berry 1977b, subsequent work in this field has concentrated on the description of chaotic eigenstates of quantum maps (Hannay 1998, Leboeuf & Voros 1990 and Leboeuf & Voros 1995) and Schanz 2005 (which contains many further references in this field). The reason for this is precisely to avoid the difficulties of coping with higher dimensions. The detailed analysis of ergodic states for $L > 1$ is still in its preliminary stages. Their Husimi functions should be concentrated in a narrow neighbourhood of the energy shell, but the often repeated statement, that this also holds for the Wigner function, is false. Indeed, it has been shown by Ozorio de Almeida *et al.* 2004 that all large scale pure states must have correspondingly small scale oscillations in their Wigner functions. Such *subplanckian*

[†] Here and throughout, we make the convenient choice that the frequency of the harmonic oscillator for the coherent states is $\omega = m = 1$.

structures (Zurek 2001) do not contribute to the averages of smooth observables (hence, the ergodicity over the energy shell). However, future refinements of experimental techniques will inevitably lead to measurable interference effects. We here introduce a conceptual tool with which to probe into the details of these higher dimensional states.

The joint Husimi-Wigner representation of quantum mechanics, **huwi representation** for short, avoids the extremes of near-classicality, or of excessive quantum complexity, that characterize alternatively its parent representations. Decomposing the phase space variables in the Wigner function, $W(\mathbf{x})$, as $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}')$, with $\mathbf{x}' = (\mathbf{x}_2, \dots, \mathbf{x}_L)$ and specifying a 2-D plane by the $2(L - 1)$ equations, $\mathbf{x}' = \mathbf{X}' = \text{const.}$, the huwi function is defined as

$$hw_{\mathbf{X}'}(\mathbf{x}_1) = (\pi\hbar)^{-(L-1)} \int d\mathbf{x}' W(\mathbf{x}_1, \mathbf{x}') \exp \frac{-(\mathbf{x}' - \mathbf{X}')^2}{\hbar}. \quad (1.2)$$

In the following section, the general features of this complete representation of the density operator in the phase space, $(\mathbf{x}_1, \mathbf{X}')$ will be discussed. However, our main focus will be centred on the huwi function for a fixed \mathbf{X}' . Viewed classically, this would be merely a slight thickening of a plane section of the Wigner function, but it is a truly quantum section, that is, for each parameter, \mathbf{X}' , $hw_{\mathbf{X}'}(\mathbf{x}_1)$ represents a different pure state. All of these states belong to the same factor Hilbert space which corresponds to the phase plane, \mathbf{x}_1 . This general scenario is developed in the following section.

In §3, we discuss the huwi representation in the simple case of coherent states and their superposition. This already exemplifies the convenient way in which the thickened section erases, not only the classical regions foreign to the section, but also all their interference effects. This is also the clearest setting in which to discuss rotations and other classical canonical transformations as tools to bring desired features into view. Van Vleck states are then analyzed in §4: It is shown that generically their huwi representation can be approximated by Gaussians in the \mathbf{x}_1 phase plane, corresponding to (squeezed) coherent states, or generalized Schrödinger cat states, centred on the discrete set of points where the constant \mathbf{X}' -plane intersects the classical surface.

Ergodic states have so far evaded any compact analytical characterization in any representation, so their study in §5 must rely on computational evidence. Given that the 2-D section of a compact $(2L-1)$ -D energy shell is a closed curve, we investigate the family resemblance between $hw_{\mathbf{X}'}(\mathbf{x}_1)$ and the eigenstates of a Hamiltonian, defined so that its level curve coincides with the section of the higher dimensional energy shell. The computational huwi patterns discussed in §6 suggest that these new kinds of pure states are characterized by dislocations in their 2-D Wigner functions that are similar to those found in the wave trains of short pulses (Nye & Berry 1974).

2. Quantum sections

Coherent states, $|\mathbf{X}\rangle$, defined in the position representation as

$$\langle q|\mathbf{X}\rangle = \left(\frac{1}{\pi\hbar}\right)^{L/4} \exp\left(-\frac{1}{2\hbar}(q-Q)^2 + \frac{i}{\hbar}P \cdot (q - \frac{Q}{2})\right), \quad (2.1)$$

form a basis for Hilbert space that is overcomplete (Glauber 1963, Sudarshan 1963, Klauder & Skagerstam 1985, Perelemov 1986, Schleich 2001 and Cohen Tannoudji *et al.* 1977): The decomposition,

$$|\psi\rangle = \frac{1}{(2\pi\hbar)^L} \int d\mathbf{X} |\mathbf{X}\rangle \langle \mathbf{X}|\psi\rangle, \quad (2.2)$$

is unique. Considering the full Hilbert space as a tensor product of factor spaces, $\mathbf{H} = \mathbf{H}_1 \otimes \dots \mathbf{H}_L \otimes \dots \mathbf{H}_L$, for each of the L degrees of freedom, it is obvious that the coherent state basis also factors, i.e.,

$$|\mathbf{X}\rangle = |\mathbf{X}_1\rangle \otimes \dots |\mathbf{X}_L\rangle \otimes \dots |\mathbf{X}_L\rangle. \quad (2.3)$$

Therefore, decomposing again $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}')$ and taking the partial overlaps,

$$|\mathbf{X}_1\rangle = \langle \mathbf{X}'|\mathbf{X}\rangle = |\mathbf{X}_1\rangle \langle \mathbf{X}_2|\mathbf{X}_2\rangle \dots \langle \mathbf{X}_L|\mathbf{X}_L\rangle \quad (2.4)$$

generates a basis for the factor space, \mathbf{H}_1 , such that each $|\psi_{\mathbf{X}'}\rangle = \langle \mathbf{X}'|\psi\rangle$ is a pure state with its wave function,

$$\langle \mathbf{X}'|\psi\rangle(q_1) = \int dq' \langle q'|\mathbf{X}'\rangle^* \langle q_1, q'|\psi\rangle. \quad (2.5)$$

The Husimi function can be defined directly in terms of coherent states, alternatively to (1.1). Given the density operator for a pure state as $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$, then

$$\mathcal{H}_\psi(\mathbf{X}) = \langle \mathbf{X}|\rho|\mathbf{X}\rangle = \text{tr } \hat{\rho}_\psi |\mathbf{X}\rangle\langle\mathbf{X}| = |\langle \mathbf{X}|\psi\rangle|^2. \quad (2.6)$$

This diagonal coherent state representation also contains full information concerning the density operator, because of the overcompleteness of the coherent state basis. The alternative definition (1.1) of the Husimi function is an immediate consequence of (Grönewold 1946, Ozorio de Almeida 1998)

$$\text{tr } \hat{\rho}_2 \hat{\rho}_1 = (2\pi\hbar)^L \int d\mathbf{x} W_2(\mathbf{x}) W_1(\mathbf{x}). \quad (2.7)$$

The partial overlap of both the bra and the ket of $\hat{\rho} = |\psi\rangle\langle\psi|$ with the same factor coherent state, $|\mathbf{X}'\rangle$, defines a reduced density operator,

$$\hat{\rho}_{\mathbf{X}'} = \langle \mathbf{X}'|\rho|\mathbf{X}'\rangle, \quad (2.8)$$

in the factor Hilbert space \mathbf{H}_1 . We also recall that the Wigner function in the full phase space is defined as (Royer 1977, Ozorio de Almeida 1998)

$$W(\mathbf{x}) = (\pi\hbar)^{-L} \text{tr } \hat{\rho} \hat{R}_{\mathbf{x}}, \quad (2.9)$$

where the operator for the reflection through the point \mathbf{x} is

$$\hat{R}_{\mathbf{x}} = 2^{-L} \int d\mathbf{Q} \left| \mathbf{q} + \frac{\mathbf{Q}}{2} \right\rangle \left\langle \mathbf{q} - \frac{\mathbf{Q}}{2} \right| e^{i\mathbf{P}\cdot\mathbf{Q}/\hbar} \quad (2.10)$$

and that similar definitions hold for Wigner functions defined in subspaces, with the appropriate adaptation of notation. Then we find that the alternative definition for the huwi function,

$$h_{\mathbf{W}_{\mathbf{X}'}}(\mathbf{x}_1) = (\pi\hbar)^{-1} \text{tr } \hat{\rho}_{\mathbf{X}'} \hat{R}_{\mathbf{x}_1} = (\pi\hbar)^{-1} \text{tr } \hat{\rho} [\hat{R}_{\mathbf{x}_1} \otimes |\mathbf{X}'\rangle\langle\mathbf{X}'|], \quad (2.11)$$

is clearly equivalent to (1.2). One can also reinterpret (2.11) as the Wigner function that represents the pure state, $\hat{\rho}_1 = \text{tr}' \hat{\rho} |\mathbf{X}'\rangle \langle \mathbf{X}'|$, where the trace is over the factor Hilbert space corresponding to the X' coordinates. If this state is represented by its Husimi function, then

$$\mathcal{H}_{X'}(\mathbf{x}_1) = \mathcal{H}(\mathbf{x}_1, \mathbf{X}'), \quad (2.12)$$

that is, the Husimi function for this quantum section is just the section of the full Husimi function. This should not be confused with the *Quantum Poincare Surface of Section* (Leboeuf & Saraceno 1990a and Leboeuf & Saraceno 1990b), which will be discussed in §5.

The definition of the huwi function is based on a wide, but rarely used freedom in the choice of representations of tensor products of Hilbert spaces. These correspond classically to Cartesian products of phase planes and it is more usual to exploit alternative generating functions for canonical transformations by exchanging variables in classical mechanics (Arnold 1978, Goldstein 1980). However, the corresponding matrix elements in quantum mechanics,

$$\langle q_1, p_2, \dots | \hat{A} | p'_1, p'_2, \dots \rangle = \text{tr } \hat{A} [|p'_1\rangle \langle q_1| \otimes |p'_2\rangle \langle p_2| \otimes \dots], \quad (2.13)$$

also form a faithful representation of the operator \hat{A} , for all choices of either p_j , or q_j , and of p'_j , or q'_j . Furthermore, it is possible to include operators $\hat{R}_{\mathbf{x}_j}$ in the same class as the dyadic operators $|p'_j\rangle \langle q_j|$ as a faithful basis for representing operators. Indeed they may be interpreted as merely defining alternative planes in the doubled phase space that corresponds to operators, just as ordinary phase space corresponds to Hilbert space (see e.g. Ozorio de Almeida 2007). Alternatively, the Husimi basis, $|\mathbf{X}_j\rangle \langle \mathbf{X}_j|$, can also be used for any of the degrees of freedom. In the case of the classical generating functions, switches of representation are usually motivated by the need to avoid singularities, i.e. caustics, in the implicit definition of the transformation. These are also a problem for semiclassical approximations to quantum evolution. In the present context, the huwi representation is singled out by the clarity with which it exhibits both classical and quantum characteristics of a pure state.

The reduced density operator, $\hat{\rho}_{\mathbf{X}'} = \langle \mathbf{X}' | \hat{\rho} | \mathbf{X}' \rangle$, resembles in some respects the partial trace, $\hat{\rho}_1 = \text{tr}' \hat{\rho}$, which is also a density operator. The corresponding Wigner function is just (Ozorio de Almeida 2007),

$$W_1(\mathbf{x}_1) = (\pi\hbar)^{-1} \text{tr } \hat{\rho} [\hat{R}_{\mathbf{x}_1} \otimes \hat{I}] = \int d\mathbf{x}' W(\mathbf{x}_1, \mathbf{x}'), \quad (2.14)$$

where \hat{I} is the identity operator. However, $\hat{\rho}_1$ will not be a pure state unless $\hat{\rho}$ can be factored into product states, whereas $\hat{\rho}_{\mathbf{X}'}$ will only be a mixed state if $\hat{\rho}$ is not itself a pure state in the full Hilbert space. The Fourier transform of $W_1(\mathbf{x}_1)$ is the chord function, or the quantum characteristic function for $\hat{\rho}_1$:

$$\chi_1(\xi_1) = \frac{1}{2\pi\hbar} \int \mathbf{x}_1 \exp\left(-\frac{i}{\hbar} \xi_1 \wedge \mathbf{x}_1\right) W_1(\mathbf{x}_1) \quad (2.15)$$

(where the skew product in the exponent is here just a plane vector product, a scalar). This can be obtained directly from the chord function, $\chi(\xi)$ for $\hat{\rho}$ as (Ozorio de Almeida 2007)

$$\chi_1(\xi_1) = (2\pi\hbar)^{2(L-1)} \chi(\xi_1, \xi' = 0), \quad (2.16)$$

that is, as a mere section of the full chord function. On the other hand, it is easy to see that the chord function for $\hat{\rho}_{\mathbf{X}'}$, i.e. the Fourier transform of $hw_{\mathbf{X}'}(\mathbf{x}_1)$ is

$$\chi_{\mathbf{X}'}(\xi_1) = (\pi\hbar)^{-(L-1)} \int d\xi' \chi(\xi_1, \xi') \exp \frac{-(\xi')^2 - i\mathbf{X}' \wedge \xi'}{\hbar}. \quad (2.17)$$

Thus, both the chord and the Wigner representations of $\hat{\rho}_{\mathbf{X}'}$ are obtained by integrating over the respective representation of $\hat{\rho}$ with a Gaussian window of width $\sqrt{\hbar}$. In contrast, we may picture these windows for the partial trace, $\hat{\rho}_1$, as infinitely wide, in the case of the Wigner function and infinitely narrow, in the case of the chord function.

If the projection of $W(\mathbf{x})$ is carried out over the momentum half space, p , instead of the components, \mathbf{x}' , we obtain the wave intensity, $|\langle q|\psi\rangle|^2$, for a pure state. Similar probability densities for linear combinations of position and momentum result from projections in other phase space directions, because of the invariance of the Wigner function with respect to linear canonical transformations, i.e. symplectic transformations. Indeed, at least for $L = 1$, it is possible to regenerate the Wigner function from a subgroup of symplectically related projections through the Radon transform (Deans 1983).

A point of practical importance concerns normalization. The definition (2.8) should be divided by $N(\mathbf{X}') = \text{tr}_1 \hat{\rho}_{\mathbf{X}'}$, so as to represent a normalized density operator. Thus, the integral of $hw_{\mathbf{X}'}(\mathbf{x}_1)$ over all \mathbf{x}_1 is $N(\mathbf{X}')$ and the corresponding chord function must have $\chi_{\mathbf{X}'}(\xi = 0) = (2\pi\hbar)^{-1}N(\mathbf{X}')$. We have left this normalization factor out of the definition, because the basic interest is in the description of the full state in the higher dimensional phase space. Indeed, for bound states there will be whole ranges of the parameters \mathbf{X}' for which the overlap of the partial coherent state with $|\psi\rangle$ will be negligible. It will be verified in the following sections that this occurs where the classical plane, $\mathbf{x}' = \mathbf{X}'$, is not even close to intersecting the classical support of $|\psi\rangle$.

So far we have not considered the possibility of squeezing the coherent states $|\mathbf{X}'\rangle$, which generate the huwi function. In the limit of infinite squeezing, these will be replaced by the position states $|Q'\rangle$, so that the equation (2.5) for the wave function in the factor space becomes simply:

$$\langle Q'|\psi\rangle(q_1) = \int dq' \langle q'|Q'\rangle^* \langle q_1, q'|\psi\rangle = \langle q_1, Q'|\psi\rangle. \quad (2.18)$$

The corresponding Wigner function is just the limiting form of the huwi function:

$$hw_{Q'}(\mathbf{x}_1) = \int d\mathbf{x}' W(\mathbf{x}_1, \mathbf{x}') \delta(q' - Q'). \quad (2.19)$$

Hence, in this limit, the factor state results from a combined momentum projection of the p' , with a (thin) section of the positions, q' . In the case of a 4-D phase space, only one variable $q' = Q'$ is fixed, so this has become a 3-D section.

3. Superpositions of coherent states

It is well known that the coherent states (2.1) have Gaussian Wigner functions (Schleich 2001, Ozorio de Almeida 1998):

$$W_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(\pi\hbar)^L} e^{-(\mathbf{x}-\mathbf{X})^2/\hbar}, \quad (3.1)$$

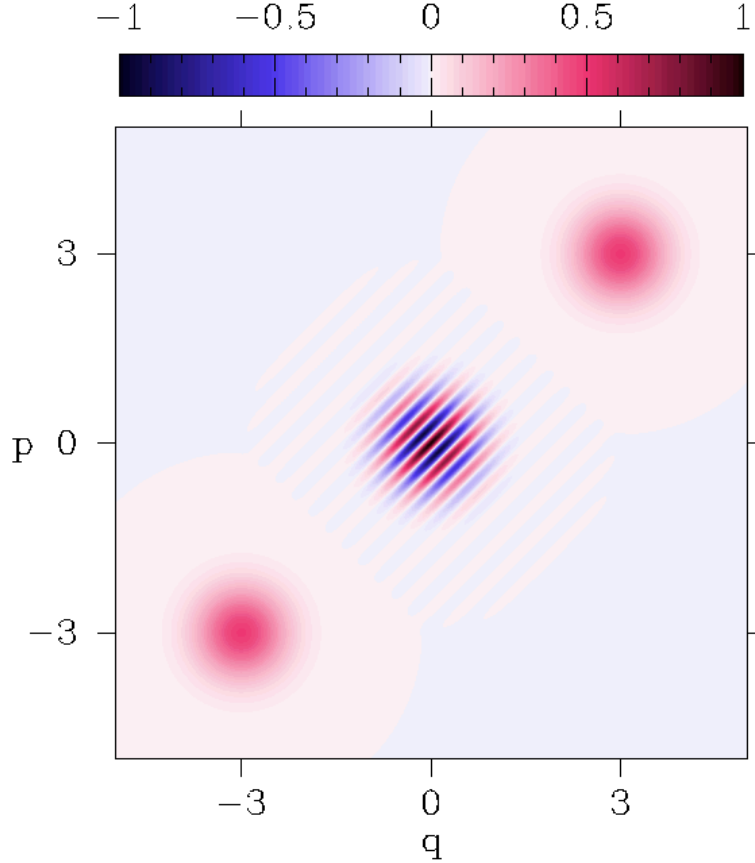


Figure 1. The Wigner function for the Schrödinger cat state displays a pair of *classical* Gaussians, one for each coherent state, and a third Gaussian modulated by interference fringes halfway between them. The chord function is a mere rescaling of the Wigner function if the midpoint lies on the origin.

whereas a superposition of coherent states, $|\mathbf{X}_a\rangle \pm |\mathbf{X}_b\rangle$, sometimes known as a *Schrödinger cat state*, has the Wigner function:

$$W_{\pm}(\mathbf{x}) = \frac{1}{[2\pi\hbar(1 \pm e^{-(\mathbf{X}_a - \mathbf{X}_b)^2/\hbar})]^L} \left[e^{-(\mathbf{x} - \mathbf{X}_a)^2/\hbar} + e^{-(\mathbf{x} - \mathbf{X}_b)^2/\hbar} \pm 2 e^{-(\mathbf{x} - (\mathbf{X}_a + \mathbf{X}_b)/2)^2/\hbar} \cos \frac{1}{\hbar} \mathbf{x} \wedge (\mathbf{X}_a - \mathbf{X}_b) \right]. \quad (3.2)$$

It consists of two *classical* Gaussians centred on \mathbf{X}_a and \mathbf{X}_b , together with an interference pattern with a Gaussian envelope centred on their midpoint, as shown in figure 1. The spatial frequency of this oscillation increases with the separation $|\mathbf{X}_a - \mathbf{X}_b|$. Increasing the number of coherent states merely increases the number of classical Gaussians and adds new localized interference patterns midway between each pair.

The overall picture does not depend on L , the number of degrees of freedom. Suppose then that L is large and that we study classical 2-D sections, $\mathbf{x}' = \mathbf{X}'$, of

the Wigner function of a superposition of a pair of coherent states that are centred on arbitrary points \mathbf{X}_j . Clearly, $W(\mathbf{x}_1, \mathbf{X}')$ will only be appreciable if the chosen \mathbf{X}' -plane is close to one of the \mathbf{X}_j' -planes on which the coherent states lie, or close to $(\mathbf{X}_j' + \mathbf{X}_k')/2$, one of their midpoints, which houses the interference pattern. There will be no doubt about the localization of a coherent state which is captured by a section close to \mathbf{X}_j' , but a section passing near a midpoint will not determine where the interference pattern is coming from. Indeed, the spatial frequency of the oscillations within the section will depend only on the projection $(\mathbf{X}_j - \mathbf{X}_k)_1$, leaving $(\mathbf{X}_j' - \mathbf{X}_k')$ completely undetermined.

There is little change between a classical section near an isolated coherent state and a quantum section, i.e. a huwi function. Indeed, integrating over the product of Gaussians in (1.2) produces a new Gaussian in the \mathbf{x}_1 plane, centred on \mathbf{X}_{a1} or \mathbf{X}_{b1} . However, if \mathbf{X}' lies near $(\mathbf{X}_a' + \mathbf{X}_b')/2$ the interference term is dampened by a factor $\exp[-(\mathbf{X}_a' - \mathbf{X}_b')^2/\hbar]$, so that it is not visible unless both section planes, \mathbf{X}_a' and \mathbf{X}_b' lie close to each other. Of course, this cancelling of interference is a familiar feature of Husimi functions and merely reflects the Husimi side of the hybrid huwi representation. The Wigner side arises if \mathbf{X}_a' and \mathbf{X}_b' are nearly the same: Then there will be practically no cancellation of the interference pattern in the \mathbf{x}_1 plane, i.e. the huwi function will resemble the classical section of the Wigner function, with two classical maxima and a central interference pattern.

It might seem too severe a restriction only to observe interferences of structures that lie on 2-D sections, but, because the Wigner function is symplectically invariant, it is always possible to picture the interference pattern between a pair of coherent states by effecting a symplectic transformation which includes them both in the same 2-D plane. On the other hand, the overlapping interference pattern for the superposition of a large number of coherent states will be vastly simplified in the huwi representation, without reaching the extreme of the Husimi function itself.

In short, a superposition of coherent states is such a simple system that it would not be worth the bother of defining the huwi function to deal with it. The purpose of this section is really to familiarize the reader with an elementary example where we already have an adequate picture. One can proceed to a more detailed investigation of superpositions of arbitrarily squeezed and rotated coherent states, but it all reduces to performing Gaussian integrals and the qualitative result is unchanged: The huwi representation displays the classical structure cut by a given 2-D plane and the interference patterns due to this 2-D structure, while washing out the interference effects of all classical structure which has not been sampled by this plane.

4. Van Vleck states

A Van Vleck wave function $\langle q|\psi\rangle$ in higher dimensions (Van Vleck 1999 and Ozorio de Almeida 1988), is supported by an L -D *Lagrangian* ψ -surface in the $(2L)$ -D phase space (Arnold 1978). Locally, each branch of this surface is obtained by the action,

$$p^j(q) = \frac{\partial S^j(q)}{\partial q}, \quad (4.1)$$

and the full wave function is then a superposition of the various branches:

$$\langle q|\psi\rangle \approx \sum_j a^j(q) \exp \left[\frac{i}{\hbar} S^j(q) \right]. \quad (4.2)$$

The amplitudes, $a^j(q)$, are determined by the classical structure and it suffices here to recall that they are finite except at caustics (generalized turning points), where the branches of the function $S^j(q)$ are joined. The semiclassical approximation to the wave function breaks down at caustics, but these can be shifted by symplectic transformations. In the case of a bound state, the ψ -surface is an L -D torus (Arnold 1978). All closed curves on the ψ -surface must satisfy approximate Bohr-Sommerfeld quantization rules, that is,

$$\oint_{\psi_1} p_1 q_1 = (n + \frac{1}{2})\hbar. \quad (4.3)$$

Thus the Van Vleck state is a generalization of simple WKB states.

The semiclassical form of the Wigner function for generalized WKB states was derived by Berry 1977a, but, rather than obtaining the huwi function by its integration with a Gaussian window, it is simpler to start from (2.5) and (4.2) to calculate the reduced wave function:

$$\begin{aligned} \langle X'|\psi\rangle(q_1) \approx \sum_j \int dq' \frac{a^j(q_1, q')}{(\pi\hbar)^{(L-1)/4}} \exp \left[-\frac{1}{2\hbar}(q' - Q')^2 - \frac{i}{\hbar} P' \cdot (q' - \frac{Q'}{2}) + \right. \\ \left. + \frac{1}{\hbar} S^j(q_1, q') \right]. \end{aligned} \quad (4.4)$$

The Gaussian factor in the integrand allows us to expand the action around Q' as

$$S^j(q_1, q') \approx S^j(q_1, Q') + p^{j'}(q_1, Q') \cdot (q' - Q'), \quad (4.5)$$

where

$$p^{j'}(q_1, Q') = \frac{\partial S^j}{\partial q'}(q_1, q' = Q') \quad (4.6)$$

Then (4.4) reduces to the Fourier integral of a Gaussian, if the slow variation of $a^j(q_1, q')$ is neglected, that is,

$$\begin{aligned} \langle X'|\psi\rangle(q_1) \approx \sum_j (4\pi\hbar)^{(L-1)/4} a^j(q_1, Q') \\ \times \exp \left[-\frac{1}{2\hbar}(p^{j'}(q_1, Q') - P')^2 - \frac{i}{2\hbar} P' \cdot Q' + \frac{1}{\hbar} S^j(q_1, Q') \right]. \end{aligned} \quad (4.7)$$

This wave function is localized in the neighbourhood of the point Q_1^j , defined by the equation, $p^{j'}(Q_1^j, Q') = P'$, that is, the position that corresponds to the intersection of the X' -plane with the ψ -surface. A linear expansion of the action, $S^j(q_1, Q')$, around this point then leads to the simplified expression:

$$\begin{aligned} \langle X'|\psi\rangle(q_1) \approx \sum_j (4\pi\hbar)^{(L-1)/4} a^j(Q_1^j, Q') \exp \left[-\frac{1}{2\hbar} P' \cdot Q' + \frac{1}{\hbar} S^j(Q_1^j, Q') \right] \\ \times \exp \left[-\frac{1}{2\hbar}(q_1 - Q_1^j) \cdot \Omega^j \cdot (q_1 - Q_1^j) + \frac{1}{\hbar} p_1(Q_1^j) \cdot (q_1 - Q_1^j) \right], \end{aligned} \quad (4.8)$$

where

$$\Omega^j(q_1^j, Q') = \frac{\partial S^j}{\partial q_1 \partial q'}(q_1^j, Q') \quad (4.9)$$

is the frequency matrix for a generalized squeezed and rotated coherent state.

Thus, the reduced wave function obtained from a Van Vleck state by a quantum section is approximately a superposition of generalized coherent states, if the section plane intersects the ψ -surface at isolated points. This is the generic situation, because the \mathbf{X}' -plane is 2-D and the ψ -surface is L -D. Thus, for $L = 2$, we have generic point intersections for continuous regions of \mathbf{X}' , whereas, for $L > 2$, the section plane may easily miss the ψ -surface, but, where intersections occur, they are generically at isolated points. The situation is quite analogous to the intersections of a given curve with a set of parallel lines in 3-D.

The huwi representation, can then generically be approximated by a Schrödinger cat state, with the generalized coherent states placed on the \mathbf{x}_1 -projection of the points of intersection of the ψ -surface with the section plane. This is similar to the states considered in the previous section. The main difference is that, for $L = 2$, variations of the \mathbf{X}' -section plane lead to similar patterns in the case of Van Vleck states, because these also intersect the ψ -surface. In contrast, the huwi representation of cat states changes drastically depending on whether the section plane comes close to any of the localized states within the full phase space. Also, it will be unusual for a higher dimensional cat to have more than one coherent state sampled by a given \mathbf{X}' -section, whereas the general huwi representation of a Van Vleck state, for $L = 2$, is a full cat with its interference patterns at all the midpoints.

It is now necessary to discuss the nongeneric case where the intersection of the \mathbf{X}' -plane and the ψ -surface is not a set of isolated points: Consider the case of a product state, $|\psi_1\rangle \otimes |\psi'\rangle$, then its classical support will be a Lagrangian surface that is a cartesian product of lower dimensional surfaces, ψ_1 -surface \times ψ' -surface. In the case of a bound state, the ψ -surface is thus factored as a $(L - 1)$ -D torus and a closed ψ_1 -curve. Then any nonempty intersection of the \mathbf{X}' -plane with the ψ -surface produces the full ψ_1 -curve (Ozorio de Almeida & Hannay 1982). All closed curves on the ψ -surface must satisfy approximate Bohr-Sommerfeld quantization rules (4.3), so that the \mathbf{X}' -section is also a quantized curve. Thus, the huwi representation of this product state is approximately the semiclassical Wigner function for $L = 1$ (Berry 1977a):

$$W_1(\mathbf{x}_1) \approx \sum_k A^k(\mathbf{x}_1) \cos \frac{S^k(\mathbf{x}_1)}{\hbar}. \quad (4.10)$$

The sum in 4.10 runs over all *chords* on the ψ_1 -curve that are centred on \mathbf{x}_1 and S_k is the area between the chord and the shell (plus a semiclassically small *Maslov phase*) as shown in figure 2. The semiclassical approximation 4.10 breaks down along *caustics*, where the amplitudes A^k , display spurious divergences. The caustics of Wigner functions are the loci of coalescing chords.

Figure 3 displays an exact Wigner function for $L = 1$, where the semiclassical structure is clearly discernible. The quantized ψ_1 -curve is a uniform maximum with a constant phase and within it there lie a succession of constant phase rings. In this region there is only one chord, $\xi(\mathbf{x})$. In the central region, the pattern becomes more complex, owing to the presence of three chords for each phase space point. It is remarkable that the relation of the interference pattern, at any point \mathbf{x} in the

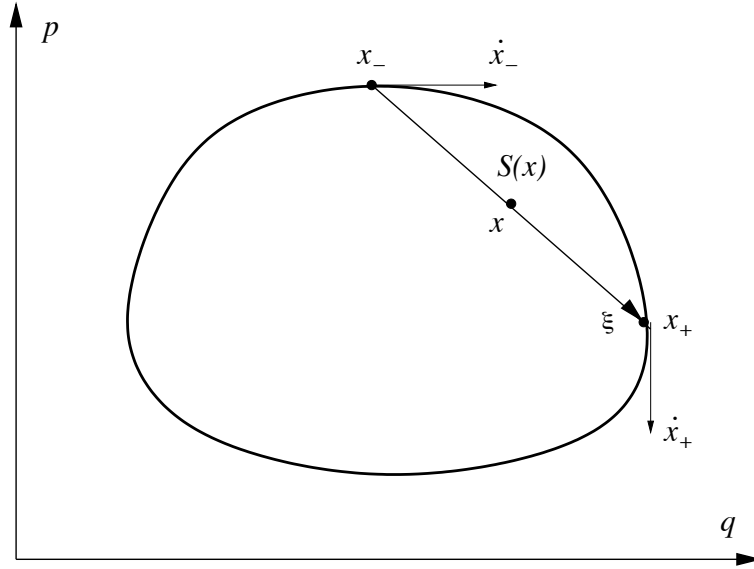


Figure 2. A single chord, ξ , is centred on \mathbf{x} if it is close to a convex quantized curve in the case of one degree of freedom. The phase of the Wigner function is proportional to the area $S(\mathbf{x})$ between the chord and the shell, while the amplitude, $A(\mathbf{x})$, depends on both phase space velocities at the tips of ξ . A caustic results from parallel tangents at the chord tips.

interior of the curve, to the chord, $\xi(\mathbf{x})$, is quite analogous to that between the interference fringes of a Schrödinger cat and the vector that separates the pair of coherent states: the analogy holds for the direction of the fringes and their spacial frequency. Indeed, it is possible to fit the semiclassical state quite accurately by a generalized cat state with a discrete set of coherent states along the quantized curve (Kenfack *et al.* 2004 and Carvalho *et al.* 2006). Evidently, the phase difference between the pair of coherent states at the chord tips must then agree with $S(\mathbf{x})$, the area between the chord and the quantized curve.

Even tori which are not products will produce huwi functions in the standard form (4.10) if this is generated by a position state, considered as the limit of the squeezed states analyzed at the end of section 2. Evidently, fixing Q' in (2.18) generates the semiclassical wave function,

$$\langle Q' | \psi \rangle(q_1) = \langle q_1, Q' | \psi \rangle \approx \sum_j a^j(q_1, Q') \exp \left[\frac{i}{\hbar} S^j(q_1, Q') \right], \quad (4.11)$$

which is also of WKB form. Moreover, all the different $(2L - 1)$ -D sections, defined by each Q' , must intersect the torus along quantized closed curves, i.e. satisfying (4.3). Therefore, each of the corresponding Wigner functions, $h\nu_{Q'}(\mathbf{x}_1)$, has the usual form (4.10).

In the following section, we display computational evidence that the huwi function of ergodic quantum states resemble in some aspects semiclassical Wigner functions of bound states of simple systems with a single degree of freedom.

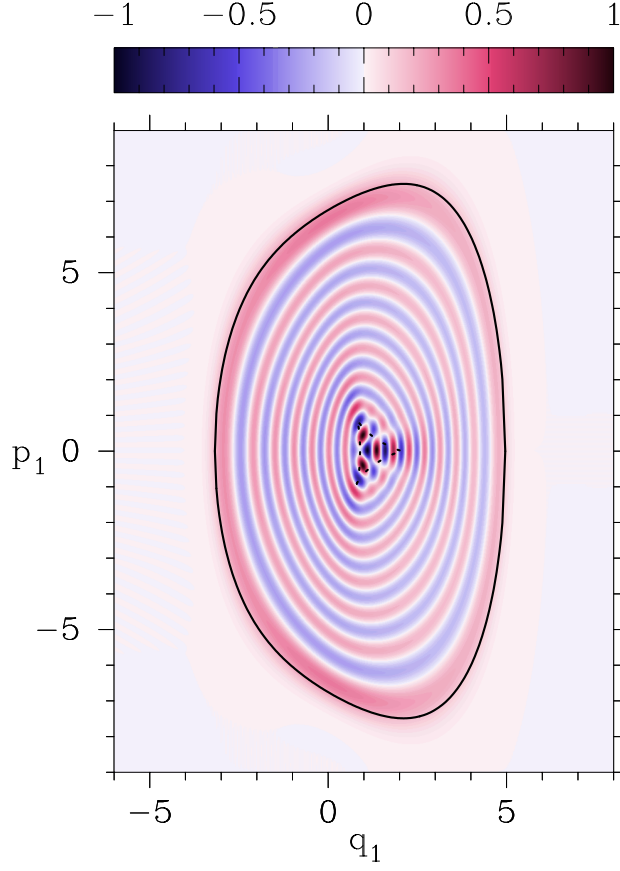


Figure 3. The Wigner function of a $L = 1$ degrees of freedom system associated with a quantized ψ_1 -curve (the full line). Inside the Wigner caustic (the dotted line) the interference pattern comes from the presence of three chords for each phase space point $\mathbf{x}_1 \equiv (p_1, q_1)$.

5. Numerical study of eigenstates of a chaotic system

The intersection of the 2-D \mathbf{X}' -plane with a $(2L - 1)$ -D compact energy shell of a bound classical system produces a closed curve in the \mathbf{x}_1 -plane, for all $L > 1$. Therefore, it is the interference of this classical structure within the section that

should determine the huwi representation for each parameter \mathbf{X}' . Viewed as a pure state for $L = 1$, this is a type of Wigner function that has not been previously studied. A resemblance to the simple semiclassical Wigner function presented at the end of the previous section may be anticipated: the chords, $\xi(\mathbf{x})$, on the section curve should lead to interference fringes parallel to the chord, just as in the WKB case. The room for freedom lies in the way this phase is shifted, as the centre \mathbf{x} is displaced. Of course, our semiclassical intuition cannot be pushed too far. The huwi function results from a *thick* quantum section, rather than a *thin* classical section, so that a neighbourhood of the full Wigner function is involved. Furthermore, we must deal with a continuum of \mathbf{X}' -parametrized Wigner functions, instead of a discrete set of Bohr-quantized states.

In the absence of a full semiclassical theory for chaotic eigenstates we resort to numerical calculations of the huwi function for a particular system. The classical *Nelson* Hamiltonian is defined as,

$$H(\mathbf{x}_1, \mathbf{x}_2) = (p_1^2 + p_2^2)/2 + 0.05 q_1^2 + (q_2 - q_1^2/2)^2, \quad (5.1)$$

and its quantum counterpart results from the replacement: $\mathbf{x}_1 \equiv (q_1, p_1) \rightarrow (\hat{q}_1, \hat{p}_1)$ and $\mathbf{x}_2 \equiv (q_2, p_2) \rightarrow (\hat{q}_2, \hat{p}_2)$ in (5.1). The restriction of the classical Hamiltonian (5.1) to the 2-D \mathbf{X}' -planes $q_1 = Q'$ and $p_1 = P'$ defines harmonic oscillators in the \mathbf{x}_2 variables: $h_2(\mathbf{x}_2) = H(\mathbf{X}', \mathbf{x}_2)$. The classical trajectories of this harmonic oscillator coincide with the intersection of the energy shell of the full Hamiltonian with the planes $q_1 = Q'$ and $p_1 = P'$. The alternative classical sections $\mathbf{x}_2 = \mathbf{X}'$ on (5.1) defines anharmonic oscillators, $h_1(\mathbf{x}_1) = H(\mathbf{x}_1, \mathbf{X}')$, whose closed trajectories are not ellipses.

The classical dynamics of this system is mixed for all energies, but we consider eigenstates $|\psi_n\rangle$ whose eigenenergies correspond to an essentially chaotic classical behavior (see Prado & de Aguiar 1994 and references therein). It would be natural to use a harmonic oscillator basis for calculating those of the Nelson Hamiltonian, but it is more efficient to use a basis of *distorted oscillators* (Toscano *et al.* 2001). In the computations we fixed $\hbar = 0.05$. In Figs.(4),(5) and (6) we show the huwi function of three different eigenstates for several constant \mathbf{x}_1 planes. It should be observed that, in spite of varying degrees of irregularity in the internal fringe pattern among these examples, the wavelength is larger for the smaller classical curves, which hence have shorter chords. In all cases the pattern decays outside the classical curve.

It is interesting to compare the huwi functions to thin sections of the full Wigner function of the eigenstates, along the planes $q_1 = Q'$ and $p_1 = P'$ as in figure 7, which also compares them to the Husimi functions for the corresponding quantum sections. These sections of the full Wigner function display interference of contributions from different parts of the energy shell outside of the plane section. Semiclassically, these oscillations can be ascribed to chords, $\xi(\mathbf{x})$, with centers, \mathbf{x} , in the section, but with tips on the energy shell far from this plane. This is particularly evident in the region $q_2 < 0$ outside the classical curve. The Gaussian smoothing (1.2), that defines the huwi function in the planes $q_1 = Q'$ and $p_1 = P'$, washed out all these interference contributions, isolating the contributions from chords with tips lying on the classical curve, or in its neighbourhood. Thus, the result is a huwi function with an interference pattern inside the classical curve and exponentially small values outside, which resembles the Wigner functions associated with a quantized curve (see figure 2). However, it is clear that the constant phase curves of the

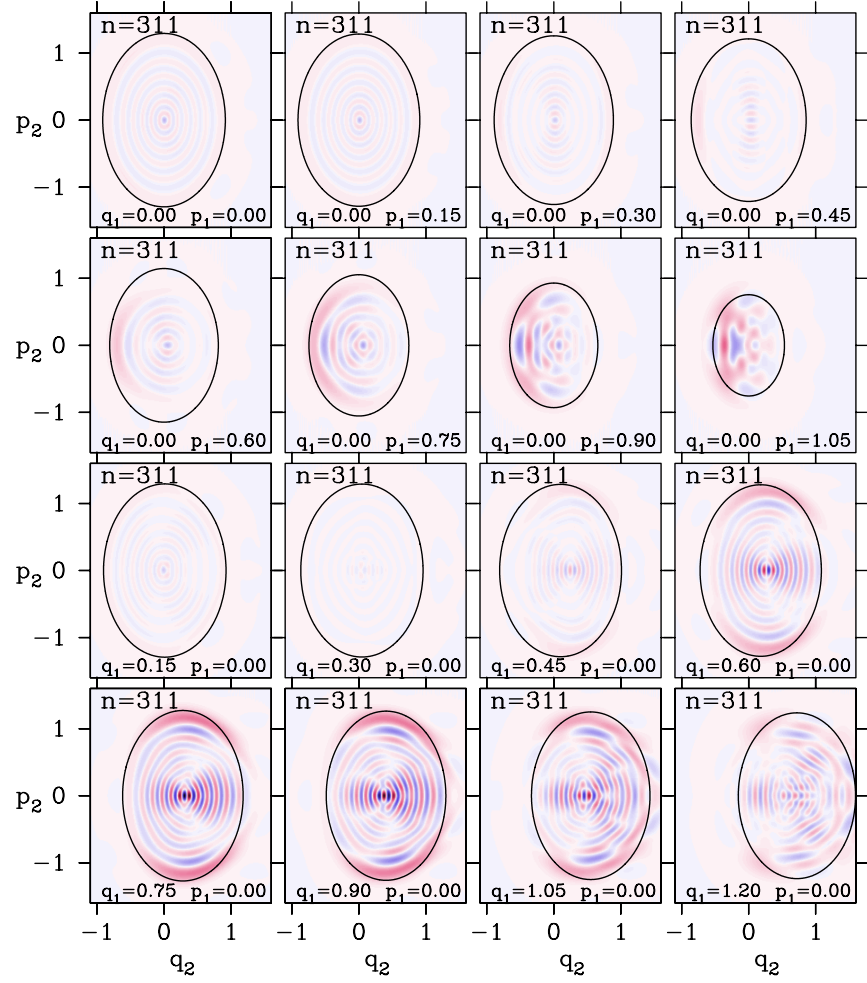


Figure 4. The huwi functions for the eigenstate, $|\psi_{n=311}\rangle$, of the *Nelson* Hamiltonian in different 2-D planes $q_1 = Q'$ and $p_1 = P'$. The black closed curve in each graph is the intersection of the energy shell with the specified 2D-plane for the eigenenergy $E_{n=311}$. This is the classical trajectory of the harmonic oscillator Hamiltonian obtained by restricting the original Hamiltonian to that plane. Relative intensities of the huwi functions are displayed for all the graphs.

interference fringes do not follow exactly the regular concentric pattern of a Wigner function for a quantized curve.

The plane $q_1 = p_1 = 0$ is special for the Nelson Hamiltonian. On the one hand, it is the symmetry plane for the reflection symmetry $q_1 \rightarrow -q_1$ of the system. Thus,

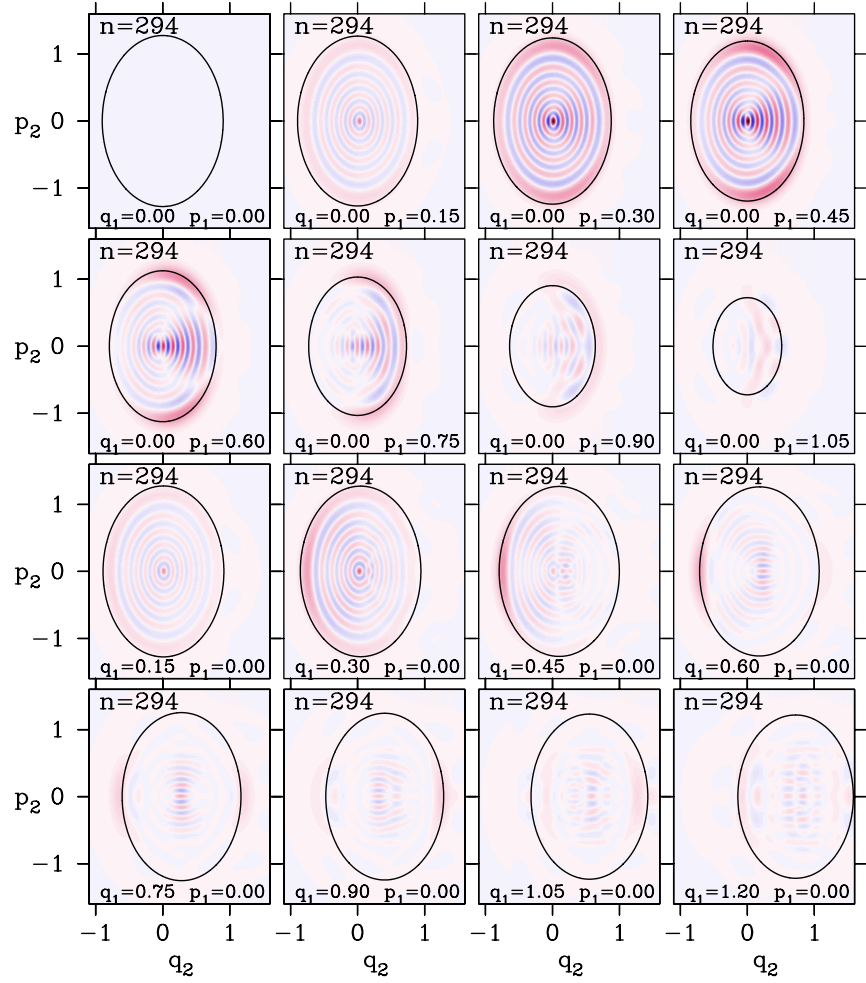


Figure 5. Idem figure 4, but for the eigenstate the *Nelson* Hamiltonian. Note the the huwi function is null, in the symmetry plane $\mathbf{x}_1 = \mathbf{X}' = (0, 0)$, because its definition involves the partial projection $\langle \mathbf{X}' | \psi_{n=294} \rangle$ of an even coherent state $|\mathbf{X}'\rangle$ with an odd $|\psi_{n=294}\rangle$ state in the \mathbf{x}_1 degree of freedom.

the huwi function in this plane discriminates the odd from the even states in the \mathbf{x}_1 degree of freedom. For odd states the huwi function is zero on the plane, because, according to (2.8), it comes from a partial projection on an even coherent state $|\mathbf{X}' = (0, 0)\rangle$. On the other hand, this is a classically invariant plane that contains the *vertical family* of periodic orbits of the full Hamiltonian. All the chords with

tips on one periodic orbit define the so called *central surface*, which, in this case, is merely the region in the invariant plane inside the classical vertical orbit (Toscano *et al.* 2001). When the classical orbit is Bohr (or "anti-Bohr") quantized, it was shown in Toscano *et al.* 2001 that the mixture of eigenstates in a narrow energy window, *i.e.* the spectral Wigner function, present a scar in the central surface of the periodic orbit, which takes the form of a pattern of concentric rings of constant phase, just like those of the semiclassical $L = 1$ Wigner function (4.10). This pattern is also evident for an individual eigenstate, as is the case for $|\psi_{n=305}\rangle$, whose eigenenergy is very close to the Bohr energy (see figure 7). In this case the huwi function in the invariant plane, as a thick section of the Wigner function, simply isolates the scar contribution of the periodic orbit (see figure 6). The thickness of the scar is sampled by the huwi functions up to a linear distance of the order $\sqrt{\hbar} \approx 0.22$ by sections parallel to the symmetry plane.

Even when the periodic orbit is far from being Bohr quantized, the huwi functions around the invariant plane isolates similar contributions from the periodic orbit, as we can see in figure 4 and 5. The similarity of these huwi functions with the Wigner functions for harmonic oscillator eigenstates is remarkable, specially if it is recalled that their energies are not Bohr-quantized.

The quantum sections which are represented by the huwi function, should not be confused with the *Quantum Poincare Surface of Section* (Leboeuf & Saraceno 1990a and Leboeuf & Saraceno 1990b). The latter is a *projection* onto a 2-D plane of the Husimi function evaluated along the 3-D energy shell. It is an invaluable tool for the study of the classical features of the quantum state. For instance, scarred states (Heller 1984, Bogomolny 1988 & Berry 1989) exhibit maxima in the neighbourhood of the points where a periodic orbit crosses the corresponding classical section (see e.g. Arranz *et al.* 2004). In contrast, the 2-D quantum section corresponds to a classical section that intersects the energy shell along a closed curve. The Husimi representation of this quantum section coincides with the section of the Husimi function (2.12) and, hence, it is entirely concentrated along the classical curve. Thus, the structure within the classical curve of the huwi function is built up of quantum interferences which are washed out in the Husimi representation of the section, as shown in figure 7.

6. Discussion

The huwi representation displays the classical structure cut by a given 2-D plane and the interference patterns due to this 2-D structure, while washing out the interference effects of all classical structure, which has not been sampled by this plane. This provides a valuable tool for the study of the eigenstates of chaotic systems, whether these be ergodic, or scarred to some extent.

The Nelson Hamiltonian, whose eigenstates were presented in the previous section, corresponds to a mixed classical system. At low energies, the motion is a perturbation of the 2-D harmonic oscillator, but becomes increasingly chaotic at higher energies. Even though the limit of truly ergodic motion is unlikely ever to be reached, it is expected that most orbits will eventually sweep over most of the energy shell, within the wavelength of the quantum motion for the states that we have considered. Therefore, we expect that most eigenstates will resemble ergodic quantum states.

Our computations clearly show that even a partial smoothing of the Wigner function is not concentrated in the neighbourhood of the energy shell. Indeed, each quantum section of the energy shell, sampled by each huwi function, marks the boundary, beyond which the huwi function is vanishingly small. But inside each section of the shell, a closed curve, the huwi function oscillates somewhat like a typical Wigner function of a Bohr-quantized state in the case of a single degree of freedom. Usually these are considered as simple examples of eigenstates of integrable systems, though it should not be forgotten that they are also trivially ergodic: the trajectories visit uniformly the entire energy shell. Therefore, the huwi representation has brought to light a natural, but unsuspected family resemblance between ergodic states of all dimensions.

The difference between huwi functions for chaotic eigenstates and simple Bohr-quantized states concerns the regularity of the wave pattern inside the classical curve. With the exception of huwi functions obtained from very symmetric sections of the energy shell, it was found in the previous section that the regular pattern of concentric wave fronts for the bohr-quantized state in figure 2 may be broken up in many places, even though the direction of the phase curves and their spacing is approximately maintained.

This scenario is curiously reminiscent of a snap shot of dislocations in wave trains, analyzed by Nye & Berry 1974. For travelling wave trains, the approximately constant wavelength is a consequence of the approximately constant temporal frequency of an initial pulse. However, the returning signal results from several scattered components with different phases. The outcome is an imperfect wave pulse with dislocations, where constant phase curves are interrupted, just as for an imperfect crystal lattice. Similar structures arise for travelling waves through spatial temporal disorder (La Porta & Surko 1996). In the present case, it is the single geometric chord centred on a given interior point of the classical curve that specifies the dominant spacial frequency near the centre, as well as the direction of the wave fronts, which must be parallel to the chord. So far, this is the same as holds for the Bohr-quantized curve, but the neighbouring oscillations of the chaotic huwi functions are closer to scattered wave trains than to regular concentric waves. This same general picture can be conjectured for 2-D quantum sections of even higher dimensional chaotic energy shells.

Perhaps, this is only a very qualitative analogy, though it indicates a direction to be pursued for the systematic characterization of huwi functions obtained from chaotic eigenstates. It is emphasised in Nye & Berry 1974 that the finite size of the pulse is a precondition for the presence of dislocations. In this respect, it should be recalled that diverse types of quantum states can be successfully fitted by localized coherent states placed along the relevant classical manifold, such that the oscillations midway between each pair of phase space Gaussians has the same spatial frequency as that of the fitted state (Kenfack *et al.* 2004 and Carvalho *et al.* 2006). One can then conjecture that the dislocations on the crests of the huwi pattern may be ascribed to the superposition of neighbouring coherent states on the classical curve with arbitrary phases.

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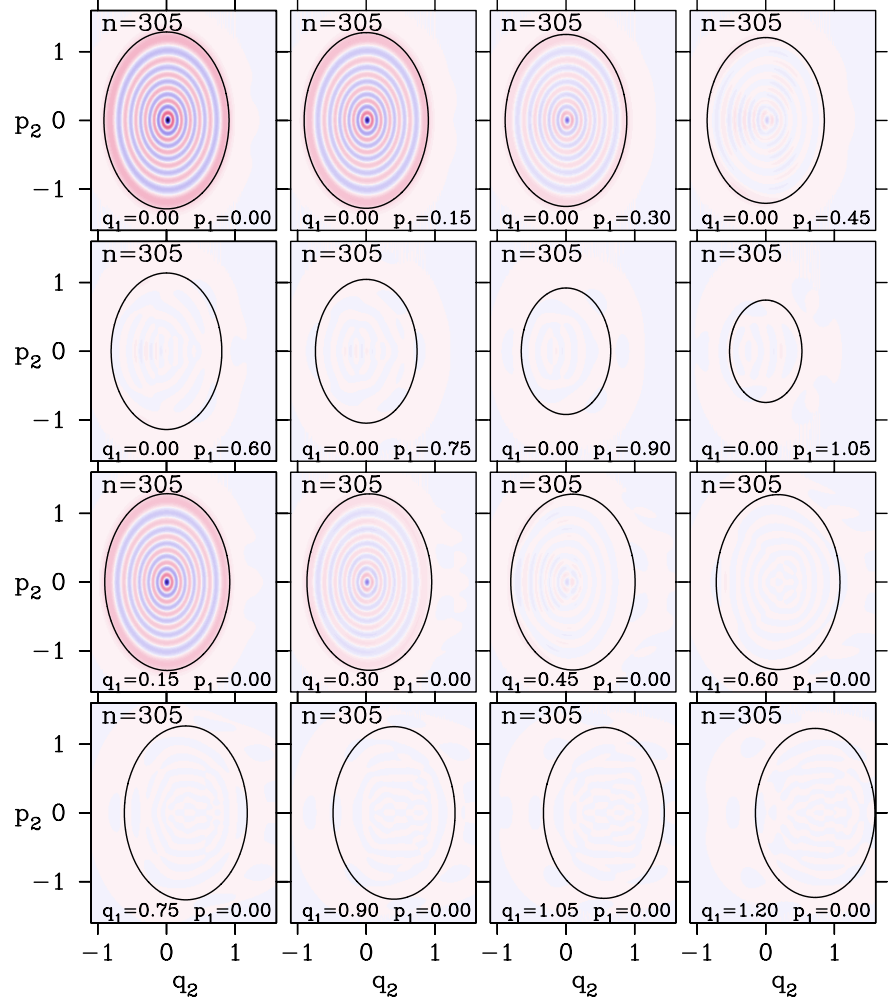


Figure 6. Idem figure 4, but for the eigenstate $n=305$. The enhanced amplitude of the huwi functions around the classically invariant plane reflects the fact that the Wigner function of this state has a scar of the periodic orbit in this plane (see figure 7). The distance of the eigenenergy $E_{n=305}$ from the Bohr energy level for the periodic orbit is of the order of a single level spacing, i.e. $\mathcal{O}(\hbar^2)$.

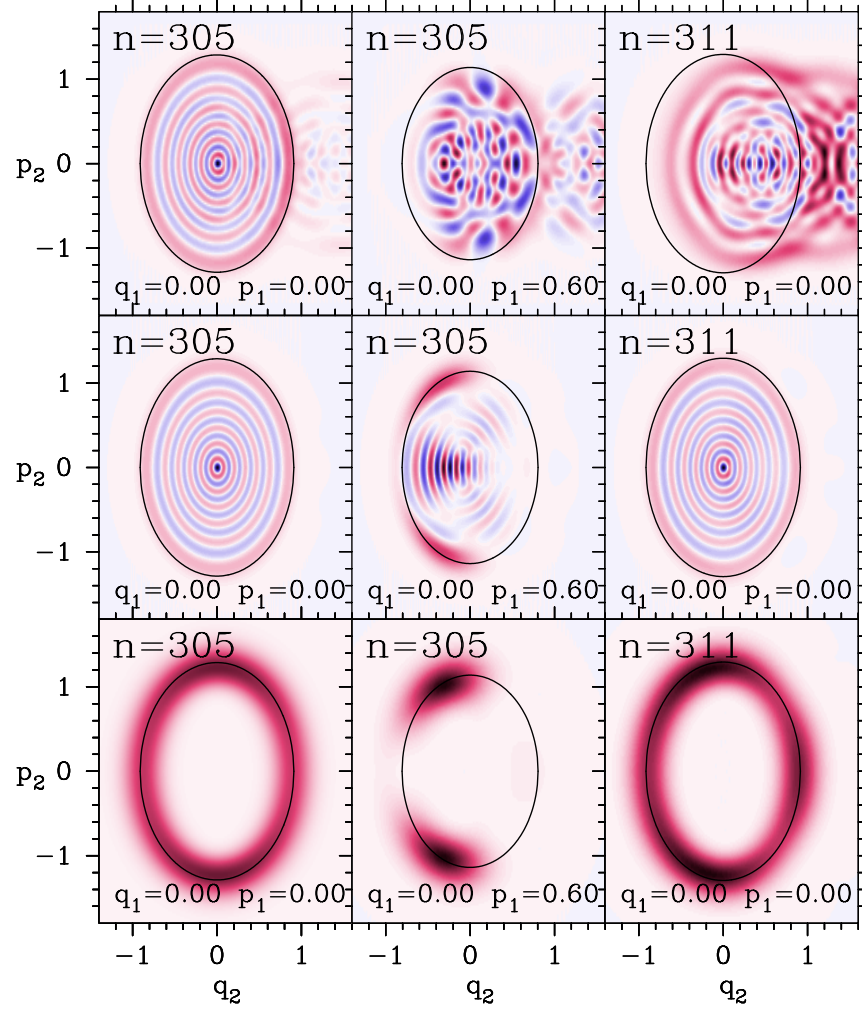


Figure 7. The top row displays plane (thin) sections of the Wigner function for three eigenstates $|\psi_n\rangle$ of the Hamiltonian (5.1). The classical curve is drawn in black, that is the energy shell in each plane $q_1 = Q'$ and $p_1 = P'$. These sections have interference contributions of different parts of the energy shell far from these plane sections (particularly evident in the region $q_2 < 0$ outside the classical curve). The middle row shows the corresponding huwi functions and the bottom row displays the sections of the Husimi functions.